APPENDIX C

Solution Techniques for Linear Algebraic Equations

C.1 CRAMER'S METHOD

Cramer's method, also known as *Cramer's rule,* provides a systematic means of solving linear equations. In practicality, the method is best applied to systems of no more than two or three equations. Nevertheless, the method provides insight into certain conditions regarding the existence of solutions and is included here for that reason.

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= f_1 \\ a_{21}x_1 + a_{22}x_2 &= f_2 \end{aligned} \tag{C.1}$$

or in matrix form

$$[A]\{x\} = \{f\} \tag{C.2}$$

Multiplying the first equation by a_{22}, the second by a_{12}, and subtracting the second from the first gives

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = f_1a_{22} - f_2a_{12} \tag{C.3}$$

Therefore, if $(a_{11}a_{22} - a_{12}a_{21}) \neq 0,$ we solve for x_1 as

$$x_1 = \frac{f_1a_{22} - f_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \tag{C.4}$$

Via a similar procedure,

$$x_2 = \frac{f_2a_{11} - f_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{C.5}$$

Note that the denominator of each solution is the same and equal to the determinant of the coefficient matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{C.6}$$

and again, it is assumed that the determinant is nonzero.

Now, consider the numerator of Equation C.4, as follows. Replace the first column of the coefficient matrix $[A]$ with the right-hand side column matrix $\{f\}$ and calculate the determinant of the resulting matrix (denoted $[A_1]$) to obtain

$$|A_1| = \begin{vmatrix} f_1 & a_{12} \\ f_2 & a_{22} \end{vmatrix} = f_1a_{22} - f_2a_{12} \tag{C.7}$$

The determinant so obtained is exactly the numerator of Equation C.4. If we similarly replace the second column of $[A]$ with the right-hand side column matrix and calculate the determinant, we have

$$|A_2| = \begin{vmatrix} a_{11} & f_1 \\ a_{21} & f_2 \end{vmatrix} = f_2a_{11} - f_1a_{21} \tag{C.8}$$

and the result of Equation C.8 is identical to the numerator of Equation C.5. Although presented for a system of only two equations, the results are applicable to any number of linear algebraic equations as follows:

Cramer's rule: Given a system of n linear algebraic equations in n unknowns x_i, $i = 1, n$, expressed in matrix form as

$$[A]\{x\} = \{f\} \tag{C.9}$$

where $\{f\}$ is known, solutions are given by the ratio of determinants

$$x_i = \frac{|A_i|}{|A|} \qquad i = 1, n \tag{C.10}$$

provided $|A| \neq 0$.

Matrices $[A_i]$ are formed by replacing the ith column of the coefficient matrix $[A]$ with the right-hand side column matrix.

Note that, if the right-hand side $\{f\} = \{0\}$, Cramer's rule gives the trivial result $\{x\} = \{0\}$.

Now consider the case in which the determinant of the coefficient matrix *is* 0. In this event, the solutions for the system represented by Equation C.1 are, formally,

$$\begin{aligned} 0x_1 &= f_1a_{22} - f_2a_{12} \\ 0x_2 &= f_2a_{11} - f_1a_{21} \end{aligned} \tag{C.11}$$

Equations (C.11) must be considered under two cases:

1. If the right-hand sides are nonzero, no solutions exist, since we cannot multiply any number by 0 and obtain a nonzero result.

2. If the right-hand sides are 0, the equations indicate that *any* values of x_1 and x_2 are solutions; this case corresponds to the *homogeneous* equations that occur if $\{f\} = \{0\}$. Thus, a system of linear homogeneous algebraic equations can have nontrivial solutions if and only if the determinant of the coefficient matrix is 0. The fact is, however, that the solutions are not just *any* values of x_1 and x_2, and we see this by examining the determinant

$$|A| = a_{11}a_{22} - a_{12}a_{21} = 0 \tag{C.12}$$

or

$$\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} \tag{C.13}$$

Equation C.13 states that the coefficients of x_1 and x_2 in the two equations are in constant ratio. Thus, the equations are not independent and, in fact, represent a straight line in the x_1x_2 plane. There do, then, exist an infinite number of solutions (x_1, x_2), but there also exists a relation between the coordinates x_1 and x_2. The argument just presented for two equations is also general for any number of equations. If the system is homogeneous, nontrivial solutions exist only if the determinant of the coefficient matrix is 0.

C.2 GAUSS ELIMINATION

In Appendix A, dealing with matrix mathematics, the concept of inverting the coefficient matrix to obtain the solution for a system of linear algebraic equations is discussed. For large systems of equations, calculation of the inverse of the coefficient matrix is time consuming and expensive. Fortunately, the operation of inverting the matrix is not necessary to obtain solutions. Many other methods are more computationally efficient. The method of Gauss elimination is one such technique. Gauss elimination utilizes simple algebraic operations (multiplication, division, addition, and subtraction) to successively eliminate unknowns from a system of equations generally described by

$$[A]\{x\} = \{f\} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} \tag{C.14a}$$

so that the system of equations is transformed to the form

$$[B]\{x\} = \{g\} \Rightarrow \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & b_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{Bmatrix} \tag{C.14b}$$

In Equation C.14b, the original coefficient matrix has been transformed to *upper triangular form* as all elements below the main diagonal are 0. In this form, the solution for x_n is simply g_n/b_{nn} and the remaining values x_i are obtained by successive back substitution into the remaining equations.

The Gauss method is readily amenable to computer implementation, as described by the following algorithm. For the general form of Equation C.13, we first wish to eliminate x_1 from the second through nth equations. To accomplish this task, we must perform row operations such that the coefficient matrix element $a_{i1} = 0$, $i = 2, n$. Selecting a_{11} as the *pivot* element, we can multiply the first row by a_{21}/a_{11} and subtract the result from the second row to obtain

$$\begin{aligned}
a_{21}^{(1)} &= a_{21} - a_{11}\frac{a_{21}}{a_{11}} = 0 \\
a_{22}^{(1)} &= a_{22} - a_{12}\frac{a_{21}}{a_{11}} \\
&\vdots \\
a_{2n}^{(1)} &= a_{2n} - a_{1n}\frac{a_{21}}{a_{11}} \\
f_2^{(1)} &= f_2 - f_1\frac{a_{21}}{a_{11}}
\end{aligned} \tag{C.15}$$

In these relations, the superscript is used to indicate that the results are from operation on the first column. The same procedure is used to eliminate x_1 from the remaining equations; that is, multiply the first equation by a_{i1}/a_{11} and subtract the result from the ith equation. (Note that, if a_{i1} is 0, no operation is required.) The procedure results in

$$\begin{aligned}
&a_{i1}^{(1)} = 0 \qquad i = 2, n \\
&a_{ij}^{(1)} = a_{ij} - a_{1j}\frac{a_{i1}}{a_{11}} \qquad i = 2, n \qquad j = 2, n \\
&f_i^{(1)} = f_i - f_1\frac{a_{i1}}{a_{11}} \qquad i = 2, n
\end{aligned} \tag{C.16}$$

The result of the operations using a_{11} as the pivot element are represented symbolically as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2^{(1)} \\ \vdots \\ f_n^{(1)} \end{Bmatrix} \tag{C.17}$$

and variable x_1 has been eliminated from all but the first equation. The procedure next takes (newly calculated) element $a_{22}^{(1)}$ as the pivot element and the operations are repeated so that all elements in the second column below $a_{22}^{(1)}$ become 0. Carrying out the computations, using each successive diagonal element as the pivot element, transforms the system of equations to the form of Equation C.14. The solution is then obtained, as noted, by back substitution

$$\begin{aligned} x_n &= \frac{g_n}{b_{nn}} \\ x_{n-1} &= \frac{1}{b_{n-1,n-1}}(g_{n-1} - b_{n-1,n}x_n) \\ &\vdots \\ x_i &= \frac{1}{b_{ii}}\left(g_i - \sum_{j=i+1}^{n} b_{ij}x_j\right) \end{aligned} \tag{C.18}$$

The Gauss elimination procedure is easily programmed using array storage and looping functions (DO loops), and it is much more efficient than inverting the coefficient matrix. If the coefficient matrix is symmetric (common to many finite element formulations), storage requirements for the matrix can be reduced considerably, and the Gauss elimination algorithm is also simplified.

C.3 *LU* DECOMPOSITION

Another efficient method for solving systems of linear equations is the so-called *LU* decomposition method. In this method, a system of linear algebraic equations, as in Equation C.14, are to be solved. The procedure is to decompose the coefficient matrix $[A]$ into two components $[L]$ and $[U]$ so that

$$[A] = [L][U] = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & U_{nn} \end{bmatrix} \tag{C.19}$$

Hence, $[L]$ is a *lower triangular matrix* and $[U]$ is an *upper triangular matrix.* Here, we assume that $[A]$ is a known $n \times n$ square matrix. Expansion of Equation C.19 shows that we have a system of equations with a greater number of unknowns than the number of equations, so the decomposition into the *LU* representation is not well defined. In the *LU* method, the diagonal elements of $[L]$ must have unity value, so that

$$[L] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ L_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & 1 \end{bmatrix} \tag{C.20}$$

For illustration, we assume a 3 × 3 system and write

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \tag{C.21}$$

Matrix Equation C.21 represents these nine equations:

$$\begin{aligned} a_{11} &= U_{11} \\ a_{12} &= U_{12} \\ a_{21} &= L_{21}U_{11} \\ a_{22} &= L_{21}U_{12} + U_{22} \\ a_{13} &= U_{13} \\ a_{31} &= L_{31}U_{11} \\ a_{32} &= L_{31}U_{12} + L_{32}U_{22} \\ a_{23} &= L_{21}U_{13} + U_{23} \\ a_{33} &= L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{aligned} \tag{C.22}$$

Equation C.22 is written in a sequence such that, at each step, only a single unknown appears in the equation. We rewrite the coefficient matrix $[A]$ and divide the matrix into "zones" as

$$[A] = \begin{array}{c} \begin{matrix} ① & ② & ③ \end{matrix} \\ \left[\begin{array}{c:c:c} a_{11} & a_{12} & a_{13} \\ \hdashline a_{21} & a_{22} & a_{23} \\ \hdashline a_{31} & a_{32} & a_{33} \end{array}\right] \end{array} \tag{C.23}$$

With reference to Equation C.22, we observe that the first equation corresponds to zone 1, the next three equations represent zone 2, and the last five equations represent zone 3. In each zone, the equations include only the elements of $[A]$ that are in the zone and only elements of $[L]$ and $[U]$ from previous zones and the current zone. Hence, the LU decomposition procedure described here is also known as an *active zone* method.

For a system of n equations, the procedure is readily generalized to obtain the following results

$$\begin{aligned} U_{1i} &= a_{1i} \qquad i = 1, n \\ L_{ii} &= 1 \end{aligned} \tag{C.24}$$

$$L_{i1} = \frac{a_{i1}}{U_{11}} \qquad i = 2, n \tag{C.25}$$

The remaining terms obtained from active zone i, with i ranging from 2 to n, are

$$L_{ij} = \frac{a_{ij} - \sum_{m=1}^{j-1} L_{im}U_{mj}}{U_{jj}} \qquad i = 2, n \quad j = 2, 3, 4, \ldots, i-1 \quad i \neq j \tag{C.26}$$

$$U_{ji} = a_{ji} - \sum_{m=1}^{j-1} L_{jm}U_{mi}$$

$$U_{ii} = a_{ii} - \sum_{m=1}^{i-1} L_{im}U_{mi} \qquad i = 2, n \tag{C.27}$$

Thus, the decomposition procedure is straightforward and readily amenable to computer implementation.

Now that the decomposition procedure has been developed, we return to the task of solving the equations. As we now have the equations expressed in the form of the triangular matrices $[L]$ and $[U]$ as

$$[L][U]\{x\} = \{f\} \tag{C.28}$$

we see that the product

$$[U]\{x\} = \{z\} \tag{C.29}$$

is an $n \times 1$ column matrix, so Equation C.28 can be expressed as

$$[L]\{z\} = \{f\} \tag{C.30}$$

and owing to the triangular structure of $[L]$, the solution for Equation C.30 is obtained easily as (in order)

$$\begin{aligned} z_1 &= f_1 \\ z_i &= f_i - \sum_{j=1}^{i-1} L_{ij}z_j \qquad i = 2, n \end{aligned} \tag{C.31}$$

Formation of the intermediate solutions, represented by Equation C.31, is generally referred to as the *forward sweep.*

With the z_i value known from Equation C.31, the solutions for the original unknowns are obtained via Equation C.29 as

$$\begin{aligned} x_n &= \frac{z_n}{U_{nn}} \\ x_i &= \frac{1}{U_{ii}}\left(z_i - \sum_{j=i+1}^{n} U_{ij}x_j\right) \end{aligned} \tag{C.32}$$

The process of solution represented by Equation C.32 is known as the *backward sweep* or *back substitution*.

In the LU method, the major computational time is expended in decomposing the coefficient matrix into the triangular forms. However, this step need be accomplished only once, after which the forward sweep and back substitution processes can be applied to any number of different right-hand forcing functions $\{f\}$. Further, if the coefficient matrix is symmetric and banded (as is most often the case in finite element analysis), the method can be quite efficient.

C.4 FRONTAL SOLUTION

The frontal solution method (also known as the *wave front solution*) is an especially efficient method for solving finite element equations, since the coefficient matrix (the stiffness matrix) is generally symmetric and banded. In the frontal method, assembly of the system stiffness matrix is combined with the solution phase. The method results in a considerable reduction in computer memory requirements, especially for large models.

The technique is described with reference to Figure C.1, which shows an assemblage of one-dimensional bar elements. For this simple example, we know that the system equations are of the form

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{C.33}$$

Clearly, the stiffness matrix is banded and *sparse* (many zero-valued terms). In the frontal solution technique, the entire system stiffness matrix is not assembled as such. Instead, the method utilizes the fact that a degree of freedom (an unknown) can be eliminated when the rows and columns of the stiffness matrix corresponding to that degree of freedom are complete. In this context, eliminating a degree of freedom means that we can write an equation for that degree of freedom in terms of other degrees of freedom and forcing functions. When such an equation is obtained, it is written to a file and removed from memory. As is shown, the net result is triangularization of the system stiffness matrix and the solutions are obtained by simple back substitution.

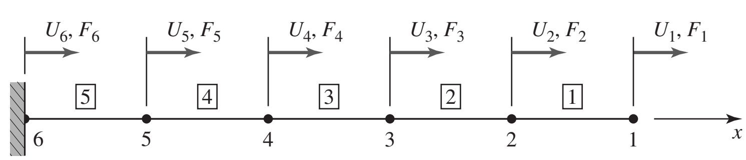

Figure C.1 A system of bar elements used to illustrate the frontal solution method.

For simplicity of illustration, let each element in Figure C.1 have characteristic stiffness k. We begin by defining a 6×6 null matrix $[K]$ and proceed with the assembly step, taking the elements in numerical order. Adding the element stiffness matrix for element 1 to the system matrix, we obtain

$$\begin{bmatrix} k & -k & 0 & 0 & 0 & 0 \\ -k & k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{C.34}$$

Since U_1 is associated only with element 1, displacement U_1 appears in none of the other equations and can be eliminated now. (To illustrate the effect on the matrix, we do not actually eliminate the degree of freedom from the equations.) The first row of Equation C.34 is

$$kU_1 - kU_2 = F_1 \tag{C.35}$$

and can be solved for U_1 once U_2 is known. Mathematically eliminating U_1 from the second row, we have

$$\begin{bmatrix} k & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_1 + F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{C.36}$$

Next, we "process" element 2 and add the element stiffness matrix terms to the appropriate locations in the coefficient matrix to obtain

$$\begin{bmatrix} k & -k & 0 & 0 & 0 & 0 \\ 0 & k & -k & 0 & 0 & 0 \\ 0 & -k & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_1 + F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{C.37}$$

Displacement U_2 does not appear in any remaining equations and is now eliminated to obtain

$$\begin{bmatrix} k & -k & 0 & 0 & 0 & 0 \\ 0 & k & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_1 + F_2 \\ F_1 + F_2 + F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{C.38}$$

In sequence, processing the remaining elements and following the elimination procedure results in

$$\begin{bmatrix} k & -k & 0 & 0 & 0 & 0 \\ 0 & k & -k & 0 & 0 & 0 \\ 0 & 0 & k & -k & 0 & 0 \\ 0 & 0 & 0 & k & -k & 0 \\ 0 & 0 & 0 & 0 & k & -k \\ 0 & 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_1 + F_2 \\ F_1 + F_2 + F_3 \\ F_1 + F_2 + F_3 + F_4 \\ F_1 + F_2 + F_3 + F_4 + F_5 \\ F_6 \end{Bmatrix} \tag{C.39}$$

Noting that the last equation in the system of Equation C.39 is a constraint equation (and could have been ignored at the beginning), we observe that the procedure has triangularized the system stiffness matrix without formally assembling that matrix. If we take out the constraint equation, the remaining equations are easily solved by back substitution. Also note that the forces are assumed to be known.

The frontal solution method has been described in terms of one-dimensional elements for simplicity. In fact, the speed and efficiency of the procedure are of most advantage in large two- and three-dimensional models. The method is discussed briefly here so that the reader using a finite element software package that uses a wave-type solution has some information about the procedure.